Lifting Böhm Trees from Terms to Term Graphs

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Outline

1. Böhm Semantics
2. Term Graph Rewriting
3. Lifting
4. Conclusion

Semantics of Term Rewriting Systems

- If a TRS is confluent and terminating then we may use normal forms.
- If a TRS is not terminating, it can still produce results. For example with the rule

  $$\mathrm{inf}(X) \to X : \mathrm{inf}(X)$$

  we have the infinite sequence

  $$\mathrm{inf}(1) \to 1 : \mathrm{inf}(1) \to 1 : 1 : \mathrm{inf}(1) \to \cdots$$

  which somehow yields

  $$1 : 1 : 1 : \cdots$$

  This result can be obtained with two theories
  - Infinitary Rewriting.
  - Böhm Semantics.

Infinite Terms

- Let $\mathcal{T}$ denote the set of terms.
- The order $\leq_\Omega$ is the transitive, reflexive, compatible closure of
$$\Omega \leq_\Omega t \in \mathcal{T}$$
- An ideal is a set I, which is
  - Downward closed: $\forall x, y : x \leq_\Omega y \wedge y \in I \Rightarrow x \in I$.
  - Directed: $\forall x, y \in I : \exists z \in I : x, y \leq_\Omega z$.
- The ideal completion of a partial order is the set of ideals order by inclusion.
- The set of infinite terms $\mathcal{T}^\infty$ is the ideal completion of $(\mathcal{T}, \leq_\Omega)$.

Direct approximation

- A function $\omega : \mathcal{T} \to \mathcal{T}^\infty$ is monotonic w.r.t. $R \subseteq \mathcal{T} \times \mathcal{T}$ if

$$\forall t_1, t_2 \in \mathcal{T} : \; t_1 \; R \; t_2 \implies \omega(t_1) \subseteq \omega(t_2)$$

- A direct approximation is a function $\omega : \mathcal{T} \to \mathcal{T}^\infty$, which is monotonic w.r.t. $\to$.
- For example

$$\begin{array}{rcl} \omega(x) & = & x \\ \omega(n) & = & n \\ \omega(\Omega) & = & \Omega \\ \omega(\mathrm{inf}(t)) & = & \Omega \\ \omega(t_1 : t_2) & = & \omega(t_1) : \omega(t_2) \end{array}$$

Böhm Semantics

- Define the *Böhm Semantics* $\mathrm{BT}_\omega : \mathcal{T} \to 2^{\mathcal{T}}$ as follows:

$$\mathrm{BT}_\omega(t) = \bigcup \{\omega(t') \mid t \twoheadrightarrow t'\}$$

- The Böhm Semantics is unique (UN_ω) if

$$\forall t_1, t_2 \in \mathcal{T} : \; t_1 = t_2 \implies \mathrm{BT}_\omega(t_1) = \mathrm{BT}_\omega(t_2)$$

  where $=$ denotes conversion.
- If the Böhm Semantics is unique then it is an infinite term:

$$\mathrm{UN}_\omega \implies \forall t \in \mathcal{T} : \; \mathrm{BT}_\omega(t) \in \mathcal{T}^\infty$$

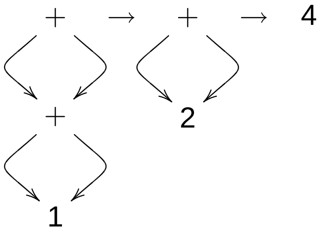
- A Böhm Tree is a unique Böhm Semantics.
- Confluence implies uniqueness:

$$\mathrm{CR} \implies \mathrm{UN}_\omega$$

Implementation of Term Rewriting Systems

$$(1+1)+(1+1) \to 2+(1+1) \to 2+2 \to 4$$

Needs 7 nodes, 6 edges and 3 steps.



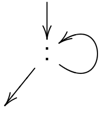
Needs 3 nodes, 4 edges and 2 steps.

$$\langle y+y \mid y=x+x, x=1\rangle \to \langle y+y \mid y=2\rangle \to 4$$

Cyclic Graphs

- Finite representation of infinite structures.

$1 : 1 : 1 : \cdots$ as



or $\langle x \mid x = 1 : x \rangle$

- Recursive functions (using lambda calculus).

$$\langle \mathrm{fib}\,10 \mid \mathrm{fib} = \lambda n.\mathrm{fib}\,(n-1) + \mathrm{fib}\,(n-2) \triangleleft n > 1 \triangleright 1 \rangle$$

where

$$x \triangleleft \mathrm{T} \triangleright y \to x \quad x \triangleleft \mathrm{F} \triangleright y \to y$$

Rewriting on Term Graphs

To implement term rewriting take a TRS and add

$$\begin{array}{lcl}
\langle C[x] \mid x = M, D\rangle & \xrightarrow[\text{es}]{} & \langle C[M] \mid x = M, D\rangle \\
\langle N \mid x = M, y = C[x], D\rangle & \xrightarrow[\text{is}]{} & \langle N \mid x = M, y = C[M], D\rangle \\
\langle N \mid x = C[x], D\rangle & \xrightarrow[\text{cs}]{} & \langle N \mid x = C[C[x]], D\rangle
\end{array}$$

$$\begin{array}{lcl}
\langle y + y \mid y = x + x, x = 1\rangle & \xrightarrow[\text{is}]{} & \langle y + y \mid y = 1 + x, x = 1\rangle \\
& \xrightarrow[\text{is}]{} & \langle y + y \mid y = 1 + 1, x = 1\rangle \\
& \xrightarrow[\text{R}]{} & \langle y + y \mid y = 2, x = 1\rangle \\
& \xrightarrow[\text{es}]{} & \langle 2 + y \mid y = 2, x = 1\rangle \\
& \xrightarrow[\text{es}]{} & \langle 2 + 2 \mid y = 2, x = 1\rangle \\
& \xrightarrow[\text{R}]{} & \langle 4 \mid y = 2, x = 1\rangle
\end{array}$$

Unwinding

- The set of cyclic terms $\mathcal{T}^\circ$ is given by

$$M ::= \langle M \mid x_1 = M_1, \cdots, x_n = M_n \rangle \mid f(M_1, \cdots, M_1) \mid x$$

- The unwinding $\mathrm{Unw}(M)$ of a cyclic term M is given as

$$\mathrm{Unw}(M) = \mathrm{BT}^{\overrightarrow{\mathrm{es}}}_{\mathrm{ext}}(M), \text{ where}$$

$$\begin{array}{lcl}
\langle C[x] \mid x = M, D \rangle & \xrightarrow[\mathrm{es}]{} & \langle C[M] \mid x = M, D \rangle \\
\mathrm{ext}(x) & = & x \\
\mathrm{ext}(f(M_1, \cdots, M_1)) & = & f(\mathrm{ext}(M_1), \cdots, \mathrm{ext}(M_1)) \\
\mathrm{ext}(\langle M \mid x_1 = M_1, \cdots, x_n = M_n \rangle) & = & \mathrm{ext}(M[x_1 := \Omega, \cdots, x_n :=
\end{array}$$

- The unwinding is unique because $\xrightarrow[\mathrm{es}]{}$ is confluent.

Problem

- Confluence is lost:

$$\begin{tikzcd}
\langle x \mid x = F(x) \rangle \arrow[r, "\mathrm{es}"] \arrow[d, "\mathrm{cs}"'] & \langle F(x) \mid x = F(x) \rangle \arrow[d, "\mathrm{cs}"'] \\
\langle x \mid x = F(F(x)) \rangle & \langle F(x) \mid x = F(F(x)) \rangle
\end{tikzcd}$$

- Adding rules solves the problem for orthogonal TRSs.
- Doesn’t work for the lambda calculus.

Problem

- Böhm Semantics proofs are easy for confluent rewrite systems.
- Graph rewriting systems are often non-confluent.
- Böhm Semantics proofs are tedious for confluent rewrite systems.

Solution

- Graph rewriting systems often consist of
  - Rules derived from a confluent TRS.
    For example
    $$\mathrm{seq}\, f\, x \to (f\, x) : (\mathrm{seq}\, f\, (f\, x))$$
    gets optimized into
    $$\mathrm{seq}\, f\, x \to \langle y : \mathrm{seq}\, g\, y \mid y = g\, x, g = f \rangle$$
  - Unwinding preserving rules.
- So we lift the the Böhm Tree for the TRS to the GRS, using infinite terms as a reference point.

Lifting a rewrite relation form terms to infinite terms

- A set $S' \subseteq S \in \mathcal{T}^\infty$ is a spanning set if

$$\forall s \in S : \; \exists s' \in S' : \; s \leq_\Omega s'$$

- We define $S[\to\rangle T$ if
  - There exists a spanning set for S, such that every element rewrites to an element in T.
  - There exists a spanning set for T, such that every element can be obtained by rewriting an element from S.
- which is equivalent to

$$\begin{array}{rl} & \forall s \in S : \exists s' \in S, t' \in T : s \leq_\Omega s' \wedge s' \to t' \\ \wedge & \forall t \in T : \exists s' \in S, t' \in T : s' \to t' \wedge t \leq_\Omega t' \end{array}$$

Example

$$A(A(X)) \rightarrow B(X)$$

$$\begin{array}{rcl}
\Omega & \twoheadrightarrow & \Omega \\
A(A(\Omega)) & \twoheadrightarrow & B(\Omega) \\
A(A(A(A(\Omega)))) & \twoheadrightarrow & B(B(\Omega)) \\
\vdots & & \vdots \\
A(A(A(\cdots))) & [\twoheadrightarrow\rangle & B(B(B(\cdots)))
\end{array}$$

Monotonicity of the Böhm Tree.

- A rewrite relation $\rightarrow$ is monotonic if

$$\forall s, s', t : \; s' \geq_\Omega s \rightarrow t \implies \exists t' : \; s' \rightarrow t' \geq_\Omega t$$

- If $\rightarrow$ is monotonic then a Böhm Tree BT_ω will be monotonic as well:

$$\forall s, t : \; s \leq_\Omega t \implies \mathrm{BT}_\omega(s) \subseteq \mathrm{BT}_\omega(t)$$

- For the remainder assume that $\rightarrow$ is monotonic and that BT_ω is a Böhm tree.

Lifting Böhm Trees from terms to infinite terms

- The direct extension is

$$\mathrm{BT}^{\infty}_{\omega}(S) = \cup\{\mathrm{BT}_{\omega}(s) \mid s \in S\}$$

- The derived extension uses the direct approximation

$$\omega^{\infty}(S) = \cup\{\omega(s) \mid s \in S\}$$

- The extensions are equivalent:

$$\mathrm{BT}^{\infty}_{\omega}(S) = \mathrm{BT}^{[\twoheadrightarrow\rangle}_{\omega^{\infty}}(S)$$

Cyclic extension

A cyclic extension $\xrightarrow[R^\circ]{} \subset \mathcal{T}^\circ \times \mathcal{T}^\circ$ of $\xrightarrow[R]{} \subset \mathcal{T} \times \mathcal{T}$ is

- Sound if

$$\forall M, N \in \mathcal{T}^\circ :\ M \xrightarrow[R^\circ]{} N \implies \mathrm{Unw}(M)[\underset{R}{\twoheadrightarrow}\rangle \mathrm{Unw}(N)$$

- Complete up to $\preceq$ if

$$\begin{array}{l} \forall M \in \mathcal{T}^\circ, s \in \mathrm{Unw}(M) :\ s \underset{R}{\twoheadrightarrow} t \implies \\ \quad \exists N \in \mathcal{T}^\circ, t' \in \mathrm{Unw}(N) :\ M \underset{R^\circ}{\twoheadrightarrow} N \wedge t \preceq t' \end{array}$$

- A typical choice for $\preceq$ is $\underset{R}{\twoheadrightarrow}$.
- For the remainder assume that $\xrightarrow[R^\circ]{}$ is a sound and complete up to $\underset{R}{\twoheadrightarrow}$ extension of $\xrightarrow[R]{}$.

Lifting Böhm Trees from terms to term graphs (1/2)

- The direct extension is

$$\mathrm{BT}^{\circ}_{\omega}(M) = \mathrm{BT}^{\infty}_{\omega}(\mathrm{Unw}(M))$$

- The derived extension uses the direct approximation

$$\omega^{\circ}(M) = \cup\{\omega(s) \mid s \in \mathrm{Unw}(M)\}$$

- The extensions are equivalent:

$$\mathrm{BT}^{\circ}_{\omega}(M) = \mathrm{BT}^{\overset{\twoheadrightarrow}{R^{\circ}}}_{\omega^{\circ}}(M)$$

Lifting Böhm Trees from terms to term graphs (2/2)

- Given a subset $\xrightarrow[R^s]{}$ of $\xrightarrow[R^\circ]{}$ and a direct approximation ω_s, such that

  $$\mathrm{BT}_{\omega_s}(M) = \mathrm{Unw}(M)$$

  define

  $$\omega_s^\circ(M) = \omega^\infty(\omega_s(M))$$

- If ω_s° is a direct approximation then

  $$\mathrm{BT}^{\underset{R^\circ}{\twoheadrightarrow}}_{\omega^\circ}(M) = \mathrm{BT}^{\underset{R^\circ}{\twoheadrightarrow}}_{\omega_s^\circ}(M)$$

Summary

- We have shown how to lift Böhm Trees from terms to graphs.
- We have not shown how to define Böhm Semantics.

Future work

Lifting Böhm Trees from dags to cyclic graphs.

- This is needed for modeling programming languages with side-effects.
- This is not yet included is current work:
  - Finite dags are CRS terms.

$$\text{let } x = M \text{ in } N \approx \mathrm{let}(M, \mathrm{in}([x](N)))$$

  - Infinite dags are not infinite CRS terms.

Example

$$\text{let } x_1 = M_1 \text{ in let } x_2 = M_2 \text{ in } \cdots f([x_1, x_2, \cdots], [x_1, x_2, \cdots])$$

is not a CRS term because we would need infinitely many abstractions above the f symbol.

- We need $f(\Omega) \leq \text{let } x = 1 \text{ in } f(x)$
- And also $d(f(\Omega), \text{let } x = 1 \text{ in } f(x)) = \frac{1}{2}$